

The tortoise coordinates and the cauchy problem in the stable study of the Schwarzschild black hole *

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Abstract

Generally, the Schwarzschild black hole was proved stable through two different methods: the mode-decomposition method and the integral method. In the paper, we show the integral method can only apply to the initial data vanishing at both the horizon and the spatial infinity. It can not treat the initial data only vanishing at the spatial infinity. We give an example to show the misleading information caused by the use of the tortoise coordinates in the perturbation equations. Subsequently, the perturbation equation in the Schwarzschild coordinates is shown not sufficient for the stable study.

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1. Introduction

In general relativity, there are two famous stationary black holes: the Schwarzschild black hole and the Kerr black hole. The Schwarzschild black hole comes from complete gravitational collapse of the massive spherically symmetric star, while the Kerr black hole results from the complete gravitational collapse of the massive spinning star. Now, the stable properties of these two black holes are crucial theoretically and astronomically in general relativity. When the black hole is proved stable, it could be candidate for the gravitational wave source. The Earth-based interferometers include LIGO, Virgo, GEO600 and TAMA, and the space-based interferometer is LISA. These interferometers all aim at the detection of the gravitational wave. The promising sources of the gravitational wave are the inspiring binaries black holes. Among these sources, the extreme-mass-ratio binaries are primary ones. The extreme-mass-ratio binary could be modelled by perturbation of a massive black hole with mass M by a small body with mass m . $\frac{m}{M}$ ranges from 10^{-1} to 10^{-9} , though m could be as large as $100M_{\odot}$. What kinds of gravitational fields could be candidate for the massive black hole? Only those whose gravitational fields are stable be qualified. Therefore, the stable study of the black holes becomes urgent now.

It is generally believed that the stable problem of the Schwarzschild black hole has been settled, while the stable properties of the Kerr black hole remain controversial and unsolved.

We have restudied the stable problem of the Schwarzschild black hole and found that its stable properties are still unsolved: they really depend on the time coordinate and where the initial time slice intersects the horizon[1]-[4].

Originally, it is in the Schwarzschild coordinate system

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\Omega^2, \quad (1)$$

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where the stable problem is studied. The well-known perturbation equations are

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r^{*2}} + VQ = 0, \quad (2)$$

is obtained in the very coordinates. In the perturbation equations (2), the tortoise coordinate r^* is defined by

$$r^* = r + 2m \ln \left(\frac{r}{2m} - 1 \right) \quad (3)$$

which approaches $-\infty$ at the horizon $r = 2m$ and ∞ at spatial infinity, and the effective potential V

$$V = \left(1 - \frac{2m}{r} \right) \left[\frac{l(l+1)}{r^2} + \frac{2m(1-s^2)}{r^3} \right] \quad (4)$$

is positive over r^* from $-\infty$ to ∞ . s could take the value 0, 1, 2 with the Eq.(2) corresponding to the scalar, electromagnetic and gravitational perturbation equations respectively. Actually, the Eq.(2) becomes the well-known Regge-Wheeler equation when $s = 2$.

Generally, researchers treated the stable problem of Eq.(2) by mode-decomposition method. Because the Eq.(2) are written by the Schwarzschild time coordinate t and r or r_* , the time coordinate t is taken for granted to define the initial time. Then the Schwarzschild black hole was proven stable[5]-[8].

Later, the integral method appeared. Researchers also investigated the stable problem by the method. In this way, they proved that any initially compact-supported perturbation data (relative to the tortoise coordinate r_*) can not blow up in the later time[6],[7]. This method confirmedly put an end to any doubts about the stable property of the Schwarzschild black hole.

We first noticed that there exists some flaws in the mode-decomposition method. Because the metric component g_{00} is zero at the horizon $r = 2m$ in the Schwarzschild coordinates, the Schwarzschild time coordinate t loses its meaning at the horizon. Actually, the future horizon corresponds to $t \rightarrow \infty$, and the past horizon corresponds to $t \rightarrow -\infty$. This shows that the coordinates (t, r) do not cover the horizons. So, t is not suitable to define the initial time containing the horizons.

The Kruskal coordinates cover the whole horizons of the Schwarzschild black hole. The two Painlevé coordinates cover the future and past horizon respectively. Correspondingly, these time coordinates are good for the investigation of the stable problem. If we use these "good" time coordinates to define the initial time, the stable properties depend complicatedly on where the initial time slice intersects with the black hole: the Schwarzschild black hole is stable when the initial time slice intersects the future horizon of the black hole, whereas the Schwarzschild black hole is unstable when the initial time slice intersects the past horizon of the black hole[1]-[4]. These extraordinary results are in striking contrast with the conclusion taken for granted that the Schwarzschild black hole is stable. They make people puzzled on the problem.

Afterward, we study the stable problem of the Rindler space time for finding the real answer to the Schwarzschild black hole's stable problem[9]. The Rindler space time has the much similar or almost the same geometrical structure with the Schwarzschild black-hole, and is mathematically simple for study of the stable problem. It is found that the same situation exists for the Rindler space time if we study the scalar field equation completely in the Rindler space time: its stable properties also depends complicatedly on where the initial time slice intersects with the Rindler space time. The Rindler space time is stable when the initial time slice intersects its future horizon, whereas it is unstable when the initial time slice intersects its the past horizon [9].

Unlike the Schwarzschild black hole, the Rindler space time is one part of the Minkowski space time, which is the most suitable and simplest space time to study. We could study the

scalar field equation completely in the Minkowski space time coordinate system and obtain definite answer to the stable problem of the Rindler space time: in this very way, it has been found that the Rindler space time is unstable[9]. This answer clearly shows which is correct among the controversial answers to the Rindler space time's stable problem, that is, the answer whose initial time slice intersects the past horizon is correct answer[9]. On the other hand, the Minkowski space time is stable, so the Rindler space time could exist as one part of the Minkowski space time.

Though we have not studied the perturbation equation completely in Kruskal coordinate system, which is too difficult to study, we still could partly infer and partly guess that the Schwarzschild black hole might be unstable by comparison with the case of the Rindler space time. We also infer that the Kruskal space time might have real entity and the Schwarzschild black hole could exist stably only as one part of the Kruskal space time.

If we insist that the stable property of the Schwarzschild black hole is not settled, we must find out why the integral method is wrong. This is main purpose of the paper.

In the paper, we show the integral method is wrong for the initial data not vanishing at the horizon in section 2. We give an example in section 3 to show the effect of the tortoise coordinates on stable study. In section 4, we also show that the perturbation equations in the Schwarzschild coordinates (tortoise coordinates) might lose some information. They wrongly lead one to believe that the wave function inside the black hole ($r < 2m$) could not influence the region outside of the black hole ($r > 2m$). So, the perturbation equations in the Schwarzschild coordinates is not sufficient for the stable study.

2. Misleading concept of compact property by the tortoise coordinates

Why researchers obtained that the Schwarzschild black hole is stable by the integral method? In the paper, we show that the tortoise properties of the Schwarzschild coordinates t and r_* are responsible for that.

Generally, the stable problem is believed to be settled by much more mathematically rigorous proof based on the integral method [5]-[8].

The perturbation equations (2) could be written as

$$\frac{\partial^2 Q}{\partial t^2} = \frac{\partial^2 Q}{\partial r_*^2} - VQ = 0, \quad (5)$$

then the operator

$$A = -\frac{d^2}{dr_*^2} + V, \quad (6)$$

is positive and self-conjugate on the Hilbert space $L^2(r_*)$ of square integral functions of r_* . It is mathematically by these very properties of the operator A and the compact-supported-ness of the quantity Q that the Schwarzschild black hole is proven stable[6]-[7]. The mathematical method is generally believed correct.

The main problem in references lies on the use of the misleading concept "compact-supported-ness of the quantity Q ": this concept is used in the tortoise coordinates t and r_* . Actually, r_* is the spatial tortoise coordinate, which changes the horizon of the Schwarzschild black hole $r = 2m$ finite into negative infinity by r_* . It is well-known that the horizon really lies in finite spatial proper distance from some point $r = r_0 > 2m$. The horizon corresponding to $r_* \rightarrow -\infty$ is only the effect of the tortoise coordinate r_* .

Really, the integral proof sounds reasonable, but it has the flaw in using the compact-supported property relative to the tortoise coordinate r_* . In fact, there are two kind of initial data. The first one vanishes both at the spatial infinity and at the horizon, which usually is denoted by the compact-supported initial data and used in the literatures. This data is really

compact-supported relative to both the coordinate r and the tortoise coordinate r_* . Because the horizon truly locates at finite position by the coordinate r , the initial data need not be zero at the horizon. There exists the second kind of the initial data, which approaches zero only at the spatial infinity. The second initial data is compact-supported only relative to the coordinate r . Therefore, only confined to the first kind of initial data, the integral method is right.

Furthermore, the Schwarzschild time coordinate t is really also a tortoise coordinate. It is also well-known that a traveller only needs finite proper time to arrive at the horizon of the Schwarzschild black hole and fall into it. In contrast to the proper time, it takes the tortoise coordinate t to infinity for the traveller to arrive at the horizon of the Schwarzschild black hole. This really means that any finite proper time process containing the horizon of the Schwarzschild black hole corresponds to $t \rightarrow \infty$ of the Schwarzschild time coordinate.

The tortoise property of the Schwarzschild time coordinate t makes it not suitable for the stable study at the horizon: for example, it may only take short proper time for some physical fields to arrive at the horizon, while it corresponds to the Schwarzschild time coordinate $t \rightarrow \infty$. In this case, the compact-supported-ness of the physical quantity Q with respect to the tortoise coordinates t and r_* are really misleading.

For example, the positive property of the operator A of Eq.(6) relies on the compact-supported-ness of the quantity Q . From Eq.(5), one obtains the following equation

$$\frac{\partial}{\partial t} \left[\int_{-\infty}^{+\infty} |\dot{Q}|^2 dr_* + \int_{-\infty}^{+\infty} Q^* A Q dr_* \right] = g(t) \quad (7)$$

where $g(t)$

$$g(t) = - \lim_{R \rightarrow \infty} \left[Q^* \frac{\partial \dot{Q}}{\partial r_*} - \dot{Q} \frac{\partial Q^*}{\partial r_*} \right]_{-R}^{+R}, \quad R \rightarrow \infty. \quad (8)$$

When $g(t) = 0$ in Eq.(7), the integral method proves that the quantity

$$\left[\int_{-\infty}^{+\infty} |\dot{Q}|^2 dr_* + \int_{-\infty}^{+\infty} Q^* A Q dr_* \right] \quad (9)$$

is constant, this in turn excludes the blowing up of the perturbation fields Q . Then, the black hole is proved stable [6]-[7]. But $g(t)$ can be non-zero. As it is just stated, the real physical process arrives at the horizon when $t \rightarrow \infty$, though it takes only finite proper time actually. So, we must study the process at the tortoise time $t = \infty + \bar{t}$ where \bar{t} is finite. Similarly, t in the Eq.(7) is also replaced by $\infty + \bar{t}$. Furthermore, the quantity Q could be non-zero at the horizon. Of course, it vanishes initially at the spatial infinity. That is $Q = 0$ at $r \rightarrow \infty$ or $r_* \rightarrow \infty$, and $Q \neq 0$ at $r = 2m$ or $r_* \rightarrow -\infty$. In this case, $g(t)$ is generally no longer zero. The mode-decomposition method is not suitable in the Eq.(7), nevertheless, we use the mode-decomposable solution $Q_k(r_*)e^{-ikt}$ as an example with $Q_k(r_*) = B_k e^{ikr_*}$ when $r_* \rightarrow -\infty$. Actually by the Eq.(8), $g(t) = 2|B_k|^2|k|^2 > 0$ is obtained for the solution. Hence the mathematical proof in references fails.

Therefore, the Reege-Wheeler equation is not suitable for mathematical proof involving the concept of the compact-supported-ness of the physical quantities. The stable proof based on this very concept is actually false, no matter how it seems mathematically rigorous.

3. The Rindler case as an example

In the following, we use the Rindler space time as an example to show why the compact-supported-ness is not suitable in the tortoise coordinates. Suppose we solve the scalar field for the Rindler space time. Because the Rindler space time is one part of the Minkowski

space time, we could write the equation in the Minkowski coordinates. The Minkowski space time's metric is

$$ds^2 = -dT^2 + dZ^2 + dx^2 + dy^2, \quad (10)$$

and the scalar field equation in the Minkowski space time is

$$-\frac{\partial^2 \Psi}{\partial T^2} + \frac{\partial^2 \Psi}{\partial Z^2} + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - \mu^2 \Psi = 0. \quad (11)$$

The Rindler space time corresponds to the part $Z > 0$, $Z^2 - T^2 > 0$ of the Minkowski space time. $Z^2 - T^2 = 0$ corresponds to the horizon of the Rindler space time. The boundaries for the Eq.(11) consist of the horizon $Z^2 - T^2 = 0$ and the spatial infinity $Z \rightarrow \infty$.

The dependence of Ψ on the variable x , y could be

$$\Psi = \psi(Z, T) e^{ik_1 x + ik_2 y} \quad (12)$$

where ψ satisfies

$$\frac{\partial^2 \psi}{\partial T^2} = \frac{\partial^2 \psi}{\partial Z^2} - (k_1^2 + k_2^2 + \mu^2) \psi. \quad (13)$$

The operator A_1 is defined as

$$A_1 = -\frac{d^2}{dZ^2} + (k_1^2 + k_2^2 + \mu^2). \quad (14)$$

Corresponding to the Minkowski coordinates T and Z , we could use the concept of the compact-supported-ness of the ψ , that is, the initial compact data of Ψ will be compact at finite time T . So we could obtain $\psi(T, Z) = 0$ as $Z \rightarrow \infty$. Because the horizon is $Z^2 - T^2 = 0$, generally, $\psi(T, Z) \neq 0$ at T . For the operator A_1 , we see that

$$\begin{aligned} I(\psi) &= \int_{horizon}^{+\infty} \psi^* A_1 \psi dZ = \int_{horizon}^{+\infty} \left[-\psi^* \frac{\partial^2 \psi}{\partial Z^2} + (k_1^2 + k_2^2 + \mu^2) \psi \psi^* \right] dZ \\ &= \left[-\psi^* \frac{\partial \psi}{\partial Z} \right]_{horizon}^{\infty} + \int_{horizon}^{+\infty} \left[\frac{\partial \psi^*}{\partial Z} \frac{\partial \psi}{\partial Z} + (k_1^2 + k_2^2 + \mu^2) \psi \psi^* \right] dZ. \end{aligned} \quad (15)$$

Because the first term in Eq.(15) could be negative for some ψ , the operator A_1 is no longer a positive operator.

For example, $I(B_0(T) e^{-kZ}) = \frac{|B_0(T)|^2}{2k} [(k_1^2 + k_2^2 + \mu^2) - k^2] e^{-2kZ_0}$ could be negative for some $k > \sqrt{k_1^2 + k_2^2 + \mu^2}$ with $Z_0 = \pm T$. This again shows that the Rindler space time is not stable.

On the other hand, we could write the scalar equation in the Rindler coordinates. The Rindler metric is

$$ds^2 = -(1 + az) dt^2 + (1 + az)^{-1} dz^2 + dx^2 + dy^2, \quad (16)$$

and the horizon is at $z = -\frac{1}{a}$. The coordinates t , z are related with T , Z by

$$T = \frac{2}{a} \sqrt{1 + az} \sinh \frac{at}{2}, \quad Z = \frac{2}{a} \sqrt{1 + az} \cosh \frac{at}{2}. \quad (17)$$

The scalar equation in the Rindler coordinates is

$$-\frac{1}{1 + az} \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial z} \left[(1 + az) \frac{\partial \psi}{\partial z} \right] - (k_1^2 + k_2^2 + \mu^2) \psi = 0. \quad (18)$$

Define the spatial tortoise coordinate z_* as

$$z_* = \frac{1}{a} \ln(1 + az) \quad (19)$$

which makes the horizon $1 + az = 0$ and the spatial infinity into $z_* \rightarrow -\infty$ and $z_* \rightarrow +\infty$ respectively. The Eq.(18) then becomes

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial z_*^2} + (1 + az)(k_1^2 + k_2^2 + \mu^2)\psi = 0. \quad (20)$$

Then the operator A_2 is

$$A_2 = -\frac{d^2}{dz_*^2} + (1 + az)(k_1^2 + k_2^2 + \mu^2). \quad (21)$$

The Eqs.(20) , (21) for the scalar field in the Rindler space time are almost the same as that in the Schwarzschild black hole (The only difference lies in the spatial infinity where the effective potential $V = (1 + az)(k_1^2 + k_2^2 + \mu^2)$ does not fall off to zero in the Rindler space time).

From above discussion, the scalar field is not compact in the tortoise coordinates t, z_* because ψ could be non-zero at the "good" finite time T on the horizons $Z^2 - T^2 = 0$. But the Eqs.(20) , (21) and the Rindler metric (16) can easily mislead one the wrong concept that ψ should be compact. Subsequently, one might wrongly prove that the Rindler space time is stable with respect to the scalar perturbation.

In the above example, we see that it is the tortoise coordinates that make us more liable to error and this error may be much more elusive to find. This tortoise property of the Schwarzschild time coordinate t has been not noticed by researchers in stable study, though it has been known in other aspects study. Actually, the influence of the tortoise property of t is much more obscure and less noticed than that of r_* .

4. Phase velocity and the cauchy problem

The Eq.(20) has another blemish that is illusive in the stable investigation of the Rindler space time. From it, we see even that the phase velocity in the direction of z axis of the scalar field can not be faster than that of the light at the future horizon. Whether or not its mass μ and wave length in other direction k_1, k_2 are zero, the the phase velocity in the direction of z axis is 1 at the horizons. The fact really makes people regard the Eq.(20) and the initial data on the hypersurface $t = \text{const}$ are a well-posed Cauchy problem in classical relativity. This very fact comes from the use of the tortoise coordinates, and is illusive. Actually, it means the scalar field in the region $Z^2 - T^2 < 0, T > 0$ can not have any influence on the Rindler space time. This is also wrong.

Though the velocity of any classical particle can not be greater than that of the light, the wave phase velocity could exceed speed limit. From the Eqs.(11), (13), we easily see that the phase velocity in the direction of Z axis could be greater than that the velocity of the light at the future horizon. So, the scalar field in the region $Z^2 - T^2 < 0, T > 0$ definitely has influence on the Rindler space time.

Similarly, we give the massive scalar perturbation equation as an example. The scalar perturbation equation in the Schwarzschild coordinates is

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r^{*2}} + \left(1 - \frac{2m}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2m}{r^3} + \mu^2\right] Q = 0 \quad (22)$$

where μ is its mass. By the geometric approximation, the scalar wave Q has the asymptotic forms

$$Q = Ae^{i\omega t \pm ikr^*} \quad (23)$$

with $\omega = k$ at the horizons. This really means that the phase velocity

$$v_p = \pm \frac{\omega}{k} = \pm 1 \quad (24)$$

is the same as that of the light at the horizon. Therefore, one gets the scalar field inside the black hole ($r < 2m$) could not influence the counterpart in the Schwarzschild black hole ($r > 2m$). This fact is truly wrong, and caused by the tortoise coordinates t, r_* . Rewriting the scalar perturbation equation in the Kruskal coordinates

$$ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}} \left[-dT^2 + dX^2 \right] + r^2 d\Omega^2, \quad (25)$$

and by the decomposition of the variables $Q = \psi(T, X)Y_{lm'}$, where $Y_{lm'}$ are the spherical harmonic functions, we obtain

$$\frac{\partial^2 \psi}{\partial T^2} - \frac{\partial^2 \psi}{\partial X^2} + \left(\frac{32m^3}{r} e^{-\frac{r}{2m}} \right) \left[-\frac{T}{2mr} \frac{\partial \psi}{\partial T} - \frac{X}{2mr} \frac{\partial \psi}{\partial X} + \mu^2 + \frac{l(l+1)}{r^2} \right] \psi = 0. \quad (26)$$

By the geometric approximation at the horizon, we obtain the asymptotic form for ψ

$$\psi = A' e^{i\omega T \pm ikX} \quad (27)$$

with the relation

$$\omega^2 = k^2 + 16m^2 e^{-1} \left[\mu^2 + \frac{l(l+1)}{4m^2} \right]. \quad (28)$$

Therefore, the phase velocity at the horizon

$$V_p = \pm \frac{\omega}{k} \quad (29)$$

is greater than $c = 1$ of the light whenever the mass μ or l is not equal to zero.

Furthermore, the second kind of data was investigated in Ref.[8], and was used to prove that the Schwarzschild black hole is stable. In the usual Penrose diagram, part *I* corresponds to our region ($r > 2m$), parts *II* and *II'* are the black-hole ($r < 2m$) and white-hole ($r < 2m$) regions respectively, and part *I'* is another region ($r > 2m$) not communicating with our region. In the proof, it was assumed that the scalar field in the regions *I'* and *II* ($r > 2m$) had no influence on the scalar field studied in the region *I*. From above discussion, the phase velocity of the Klein-Gordon equation in the Kruskal coordinates is greater than that of the light even at the horizon. This in turn influences the propagation of the scalar fields. So the proof of the theorems in Ref.[8] is not guaranteed (The causal property can not guarantee that " φ' " coincides with φ in the intersection of the region *I* with the future of C'^1).

The method in references really applies only to perturbations of the Schwarzschild black hole that vanish at the bifurcation surface. It can not treat the perturbations with initial non-zero data at the horizon. In addition, the perturbation equations in the Schwarzschild coordinates mislead one and are not sufficient for the stable study.

Therefore the stable properties of the Schwarzschild black-hole still remain unsettled now, and we only partly infer that it is unstable by comparison with that in the Rindler space time. On the other hand, the Kruskal space time might be stable and have physical entity in comparison with that in the Rindler space time. So the Schwarzschild black-hole could be stable only as one part of the Kruskal space time[9].

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¹see Ref.[8] page 895

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